Backpaper Exam - Optimization
B. Math II

31 January, 2022

> (i) Duration of the exam is 3 hours.
>
> (ii) The maximum number of points you can score in the exam is 80 (the final score willl be based on the rules of backpaper exam.)
>
> (iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

   (a) (5 points) Compute the singular value decomposition of A.

   (b) (10 points) Find a vector $\vec{x} \in \mathbb{R}^4$ which minimizes $\|A\vec{x} - \vec{b}\|_2$ (with justification).

Total for Question 1: 15

2. Consider the following linear programming problem:

$$\text{Minimize } x_1 + \alpha x_2 + x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &\geq 0 \\ -x_1 + x_3 &\geq 0 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

(a) (5 points) Solve the above LPP for $\alpha = -1$.

(b) (10 points) Which values of α lead to an unbounded LPP?

Total for Question 2: 15

3. The following tableaux corresponds to an *optimal basis* for a linear programming problem, where x_1, x_2, x_3 are the original primal variables, and s_1 and s_2 are the slack variables corresponding to the two constraints.

	x_1	x_2	x_3	s_1	s_2
35	0	2	0	3	2
$\frac{5}{2}$	0	$\frac{1}{4}$	1	$\frac{1}{2}$	0
$\frac{5}{2}$	1	$-\frac{1}{2}$	0	$-\frac{1}{6}$	$\frac{1}{3}$

(a) (10 points) Deduce the original linear program with justification.

(b) (4 points) Write the dual of the original program.

(c) (6 points) Explain why the reduced costs of the slack variables in the tableau are equal to the negative of values of the dual optimal variables for the corresponding constraints.

(d) (4 points) From the given tableax, determine B^{-1}, where B is the given optimal basis.

(e) (6 points) Consider adding a new variable x_4 to the original problem with corresponding column $\begin{bmatrix}1\\1\end{bmatrix}$ in the constraint matrix, and (unknown) cost coefficient c_4. Calculate the (expanded) tableau corresponding to B, the given basis, by calculating the entries for the new column corresponding to x_4 (the reduced cost in row zero must be expressed as a function of c_4, the unknown cost coefficient).

(f) (5 points) Determine for what values of c_4 the current basis B remains optimal.

Total for Question 3: 35

4. For non-zero vectors $\vec{a}_i \in \mathbb{R}^n, b_i \in \mathbb{R}$ $(1 \leq i \leq m)$, consider the piecewise-linear minimization problem,

$$\underset{\vec{x}\in\mathbb{R}^n}{\text{minimize}} \max_{1\leq i\leq m} (\langle \vec{a}_i, \vec{x}\rangle + b_i).$$

Let A be the $m \times n$ matrix with rows $\vec{a}_i^T$, and $\vec{b}$ be the vector in $\mathbb{R}^m$ with components $b_1, \ldots, b_m$.

(a) (5 points) Derive a dual problem, based on the Lagrange dual of the equivalent problem

$$\underset{\vec{x}\in\mathbb{R}^n, \vec{y}\in\mathbb{R}^m}{\text{minimize}} \max_{1\leq i\leq m} y_i$$

subject to

$$A\vec{x} + \vec{b} = \vec{y}.$$

(b) (5 points) Let α^* be the optimal value for the above optimization problem. Let $\vec{p} \in \mathbb{R}^m$ be a probability vector, that is, $\vec{p} \geq \vec{0}$ and $\sum_{i=1}^m p_i = 1$, such that $\vec{p}^T A = \vec{0}^T$. Show that $\langle \vec{p}, \vec{b} \rangle \leq \alpha^*$.

(c) (5 points) In order to find the best possible lower bound of the form described in part (b), we form the linear programming problem

$$\underset{\vec{p} \in \mathbb{R}^m}{\text{maximize}} \langle \vec{p}, \vec{b} \rangle$$

subject to

$$\vec{p}^T A = \vec{0}^T, \; \sum_{i=1}^{m} p_i = 1, \vec{p} \geq \vec{0}.$$

Show that the optimal value of this problem is equal to α^*.

Total for Question 4: 15